

1) Soient $a, b \in \mathbb{R}$ et $x \in]0; +\infty[$.

$$\frac{a}{x} + \frac{bx}{x^2+1} = \frac{a(x^2+1)}{x(x^2+1)} + \frac{bx^2}{x(x^2+1)}$$

$$\frac{a}{x} + \frac{bx}{x^2+1} = \frac{a(x^2+1) + bx^2}{x(x^2+1)}$$

$$\frac{a}{x} + \frac{bx}{x^2+1} = \frac{ax^2 + a + bx^2}{x(x^2+1)}$$

$$\frac{a}{x} + \frac{bx}{x^2+1} = \frac{(a+b)x^2 + a}{x(x^2+1)}$$

Donc par identification :

$$\begin{cases} a+b=0 \\ a=1 \end{cases} \Leftrightarrow \begin{cases} b=-1 \\ a=1 \end{cases}$$

Donc $\frac{1}{x(x^2+1)} = \frac{1}{x} - \frac{x}{x^2+1}$

2) Posons $u' = \frac{x}{(x^2+1)^2}$ donc $u = \frac{-1}{2(x^2+1)}$

$v = \ln(x)$ donc $v' = \frac{1}{x}$

donc $\int_a^1 \frac{x \ln x}{(x^2+1)^2} dx = \left[\frac{-\ln x}{2(x^2+1)} \right]_a^1 - \int_a^1 \frac{-1}{2x(x^2+1)} dx$

$$\int_a^1 \frac{x \ln x}{(x^2+1)^2} dx = \left[\frac{-\ln x}{2(x^2+1)} \right]_a^1 + \frac{1}{2} \int_a^1 \frac{1}{x(x^2+1)} dx$$

$$\int_a^1 \frac{x \ln x}{(x^2+1)^2} dx = \frac{\ln a}{2(a^2+1)} + \frac{1}{2} \int_a^1 \frac{1}{x(x^2+1)} dx$$

Étudions maintenant $\int_a^1 \frac{1}{x(x^2+1)} dx$.

Avec la question 1 :

$$\int_a^1 \frac{1}{x(x^2+1)} dx = \int_a^1 \frac{1}{x} - \frac{x}{x^2+1} dx$$

$$\int_a^1 \frac{1}{x(x^2+1)} dx = \left[\ln x + \frac{1}{2(x^2+1)} \right]_a^1$$

$$\int_a^1 \frac{1}{x(x^2+1)} dx = \frac{1}{4} - \ln a - \frac{1}{2(a^2+1)}$$

Donc

$$\int_a^1 \frac{x \ln x}{(x^2+1)^2} dx = \frac{\ln a}{2(a^2+1)} + \frac{1}{2} \int_a^1 \frac{1}{x(x^2+1)} dx$$

$$\int_a^1 \frac{x \ln x}{(x^2+1)^2} dx = \frac{\ln a}{2(a^2+1)} + \frac{1}{2} \left(\frac{1}{4} - \ln a - \frac{1}{2(a^2+1)} \right)$$

$$\int_a^1 \frac{x \ln x}{(x^2+1)^2} dx = \frac{\ln a}{2(a^2+1)} + \frac{1}{8} - \frac{\ln a}{2} - \frac{1}{4(a^2+1)}$$

$$\int_a^1 \frac{x \ln x}{(x^2+1)^2} dx = \ln a \left(\frac{1}{2(a^2+1)} - \frac{1}{2} \right) + \frac{1}{8} - \frac{1}{4(a^2+1)}$$

$$\int_a^1 \frac{x \ln x}{(x^2+1)^2} dx = \ln a \left(\frac{1}{2(a^2+1)} - \frac{1}{2} \right) + \frac{1}{8} - \frac{1}{4(a^2+1)}$$

$$\int_a^1 \frac{x \ln x}{(x^2+1)^2} dx = \ln a \left(\frac{-a^2}{2(a^2+1)} \right) + \frac{1}{8} - \frac{1}{4(a^2+1)}$$

$$\int_a^1 \frac{x \ln x}{(x^2+1)^2} dx = \frac{-a^2 \ln a}{2(a^2+1)} + \frac{1}{8} - \frac{1}{4(a^2+1)}$$

3) $\lim_{x \rightarrow 0} K(a) = \frac{-0}{2(0^2+1)} + \frac{1}{8} - \frac{1}{4(0^2+1)} = \frac{-1}{8}$ d'après les croissances comparées et les sommes et quotients de limites.

Donc $\int_0^1 \frac{x \ln x}{(x^2+1)^2} dx = \frac{-1}{8}$.

4) En refaisant l'intégration par parties on obtient :

$$K(b) = \frac{-2 \ln b + 2b^2 \ln b + 2 \ln b - b^2 \ln(b^2+1) - \ln(b^2+1) + b^2 \ln 2 + \ln 2}{4(b^2+1)}.$$